UDC 538.4

MODELS OF POLARIZABLE MEDIA AND THE AVERAGED RELATIONS WHICH CORRESPOND TO THESE IN THE CASE OF HIGH-FREQUENCY ELECTROMAGNETIC FIELD

PMM Vol. 41, № 2, 1977, pp. 271-281

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(Received August 12, 1976)

Models of polarizable continuous media are considered on the basis of a universal variational equation [1]. The aim is to obtain a reasonably full information about the behavior of media in high-frequency electromagnetic field, with the use of simplest assumptions of a macroscopic character on energy dissipation and on the form of thermodynamic functions. The proposed model differs from models in which the dependence of the polarization vector on the electric field is single-valued, first, in that besides reversible components of the polarization vector, their irreversible components on which free energy depends and whose variation is associated with energy dissipation are considered. Second, it differs by the presence of three-dimensional gradients and time derivatives of the polarization vector among the arguments of free energy. The introduction of irreversible components of the polarization vector is similar to the transition in the theory of elasticity to simplest models of viscoelastic media. The related complication can be readily overcome by introducing in the analysis a supplementary set of internal parameters, as was originally done by Biot in the case of viscoelasticity [2].

The construction of models is based on the variational equation of the form used in [1, 3], from which with allowance for the second law of thermodynamics and the use of traditional assumptions about parameters that define irreversibility, a closed system of equations is obtained in classical approximation. Conversion to averaged formulas which describe the interaction between field and medium is considered in the case of high-frequency electromagnetic field.

Many publications dealing with the construction of models of continuous media with allowance for polarization and magnetization, among which [4 — 7] should be noted, are from the methodological point of view close to this paper. Gyromagnetic effects are taken into consideration in [5, 6], while the case of ferromagnetic materials susceptible to magnetic hysteresis is investigated in [7]. Certain peculiarities of permittivity dispersion were not described in those papers. Since the dispersion occurs when the electric and magnetic fields alter at high frequency, hence their results must be extended in this case. Because electromagnetic waves at high field intensities are at present technically obtainable, it is interesting to construct closed models that would define the interaction between a high-frequency field and a medium.

1. The controlling parameters. We introduce in the three-dimensional Euclidean space the observer's inertial coordinate system with coordinates x^α and an accompanying system with coordinates ξ^α. Greek letter superscripts run everywhere through the values 1, 2, 3, and t is the time in the observer's system. To each point of the moving medium is attached its own system of coordinates. Components of vectors of electrical intensity and magnetic induction fields are denoted, respectively, by $E^{*\alpha}$ and $B^{*\alpha}$. These vectors are determined in their own system of coordinates and then transferred to the observer's coordinate system by conventional formulas for three-dimensional vector transformation. Components of these vectors, denoted in the observer's inertial system by E^α and B^α, are related to components $E^{*\alpha}$ and $B^{*\alpha}$ by the transformation formulas derived in [4] which contain the velocity of medium.

In the final results we shall retain only terms of order v / c, where v is the velocity of the medium and c is the speed of light in vacuum, however in the derivation of Euler's equations from the basic variational equation it is necessary to use relativistic transformation formulas with terms of order v^2 / c^2. In what follows the asterisk indicates that the related quantity is determined in the attached coordinate system. If the corresponding components are the same in the attached and the observer's systems within the stipulated accuracy, the asterisk is omitted. The components E_α and B^α are expressed in terms of components of vector potential A_α and the scalar potential by formulas

$$E_\alpha = -\nabla_\alpha\varphi - c^{-1}\partial A_\alpha / \partial t, \quad B^\alpha = \varepsilon^{\alpha\beta\gamma}\nabla_\beta A_\gamma$$

where $\varepsilon^{\alpha\beta\gamma}$ are components of completely antisymmetric Levi-Chivita pseudo-tensor.

Various internal parameters, in particular those associated with the irreversibility of polarization and magnetization processes, may be introduced in the analysis to determine with a suitable selection of thermodynamic functions the highly complex character of relaxation of polarization and magnetization. Here it is assumed that there is only one such (vectoral) parameter π^* with components $\pi^{*\alpha}$, which is associated with the irreversibility of the polarization process. It is shown below that for some particular specification of free energy that parameter has the meaning of the vector of irreversible polarization of a unit of mass determined in the attached system of coordinates.

We shall define the deformation process by the components of the deformation tensor $\varepsilon_{\alpha\beta}$, in particular, the dependence of deformations may be determined in terms of density ρ. We introduce the following notation: v^α for the components of the medium velocity vector, K_B for constants of physical and geometrical character which are assumed to be functions of only the attached coordinates, T for the absolute temperature, ρ_e for the unit volume charge, and i_α for components of the conductance current.

2. The variational equation. In what follows, variations of the unknown quantities x^α, A_α, φ, $\pi^{*\alpha}$ and T^* which are considered in constant concomitant coordinates ξ^β and at the characteristic time associated with each particle, are considered as independent, $\delta K_B = 0$ by definition, and variations of remaining quantities are expressed in terms of basic quantities in conformity with appropriate formulas. The corresponding procedure is presented in [1, 3].

Construction of the model is based on the variational equation of the form

$$\delta \int_{V_4} \Lambda d\tau_4 + \delta W^* + \delta W = 0 \tag{2.1}$$

where Λ is the combined Lagrangian of the electromagnetic field and medium, δW^* is some given functional, whose selection will be described below, δW is a functional determinable on the three-dimensional boundary of the four-dimensional volume V_4 by the specified Λ and δW^*

For the definition of the Lagrangian of the electromagnetic field and medium we use the formula

$$\Lambda = \frac{1}{8\pi}(E_\alpha E^\alpha - B_\alpha B^\alpha) + \rho(K - F)$$

where $K = v^2 / 2$ is the kinetic energy of the medium mass unit and F is the free energy of a unit of mass which for a particular model is specified as a function of the metric tensor components in the observer's system $g_{\alpha\beta}$ and of subsequent vector and tensor components and, also, of scalars $E^{*\alpha}$, $B^{*\alpha}$, $\varepsilon_{\alpha\beta}^*$, $\pi^{*\alpha}$, $\nabla_\alpha^*\pi^{*\beta}$, $d^*\pi^{*\alpha} / dt$, T^* and K_B^*, with all these arguments determined in the attached system of coordinates and transformed to the observer's system. The symbols d^* / dt and ∇_α^* denote differentiation with respect to time and the covariant differentiation with respect to the coordinate in the attached system, respectively. With such selection of arguments the free energy F of a unit of mass is evidently such, that it is invariant with respect to various inertial reference systems.

It is shown below that the obtained equations are also invariant with respect to the Lorentz transformation within the specified here accuracy (within terms of order v / c). Assuming that the energy exchange between the field and medium is determined by the release of Joule heat and by the polarization process, we specify the entropy balance as follows:

$$\rho T \frac{d^*S}{dt} = \tau^{*\alpha\beta}\nabla_\beta^* v_\alpha + E_\alpha^* i^{*\alpha} + Q_\alpha^* \frac{d^*\pi^{*\alpha}}{dt} + \frac{dQ^e}{dt} \tag{2.2}$$

where S is the entropy of a unit of mass, $\tau^{*\alpha\beta}$ are components of viscous stresses, Q_α^* are components of the generalized force which determines the irreversibility of the polarization process, and dQ^e / dt is the rate of external heat influx, except the heat from the electromagnetic field, to a unit of the medium volume. All quantities appearing in this formula are determined in the attached coordinate system and transformed to that of the observer. If the heat influx dQ^e is due only to the heat conduction process, $dQ^e = -\nabla_\alpha^* q^{*\alpha} dt$ and the formula for the internal growth of entropy in conformity with the basic assumption is of the form

$$\rho T \frac{d_iS}{dt} = \tau^{*\alpha\beta}\nabla_\beta^* v_\alpha + E_\alpha^* i^{*\alpha} + Q_\alpha^* \frac{d^*\pi^{*\alpha}}{dt} - \frac{\nabla_\alpha^* T}{T} q^{*\alpha} \tag{2.3}$$

and the inequalities $d_iS \geqslant 0$ and $d_iS > 0$ hold, respectively, for all processes and for irreversible processes, respectively.

In the case of continuous motion and in conformity with the second law of thermodynamics the functional δW^* is specified in the form

$$\delta W^* = \int_{V_4} (-\rho S\delta T^* - \tau_\alpha^{*\beta}\nabla_\beta^*\delta x^\alpha + F_\alpha \delta x^\alpha - Q_\alpha^* \delta\pi^{*\alpha} +$$
$$c^{-1} j^\alpha \delta_L A_\alpha - \rho_e \delta_L \varphi)\, d\tau_4$$
$$\delta_L A_\alpha = \delta A_\alpha + A_\beta \nabla_\alpha \delta x^\beta, \quad j^\alpha = i^\alpha + \rho_e v^\alpha$$
$$\delta_L \varphi = \delta\varphi - c^{-1} A_\beta \partial\delta x^\beta / \partial t$$

where F_α are components of the vector of volume forces.

3. The system of equations. In accordance with the assumptions made about the form of Lagrangian Λ and of functional δW^* we obtain from the variational equation (2.1) the following system of equations:

$$\varepsilon^{\alpha\beta\gamma}\nabla_\beta H_\gamma = \frac{1}{c}\frac{\partial D^\alpha}{\partial t} + \frac{4\pi}{c} j^\alpha, \quad \nabla_\alpha D^\alpha = 4\pi\rho_e \tag{3.1}$$

$$S = -\frac{\partial F}{\partial T^*}, \quad \rho a_\alpha = \nabla_\beta{}^* p_\alpha{}^\beta + R_\alpha + F_\alpha$$

$$\frac{\partial F}{\partial \pi^{*\alpha}} - \frac{1}{\rho}\nabla_\beta{}^*\left[\rho\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\alpha})}\right] - \frac{d^*}{dt}\left[\frac{\partial F}{\partial(d^*\pi^{*\alpha}/dt)}\right] + \frac{1}{\rho}Q_\alpha{}^* = 0$$

where the first two equations represent the second pair of Maxwell equations (the first pair is the corollary of the assumption about the existence of scalar and vector potentials), the third equation determines the entropy, the fourth is the equation of momenta, and the last is an additional dynamical equation which links vector π^* with the electric field intensity and other parameters. The following notation is used: a_α for components of the vector of medium acceleration, and for the components of stress tensor $p_\alpha{}^\beta$, volume force R_α, and also for vectors of electrical induction D^α and for the magnetic field intensity H_γ we have

$$p_\alpha{}^\beta = \rho\frac{\partial F}{\partial \varepsilon^*_{\nu\beta}} g_{\nu\alpha} - \rho\varepsilon_{\nu\alpha}\frac{\partial F}{\partial \varepsilon^*_{\beta\nu}} - \rho\varepsilon_\nu{}^\beta\frac{\partial F}{\partial \varepsilon^*_{\nu\gamma}} g_{\gamma\alpha} - \rho\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\gamma})}\nabla_\alpha{}^*\pi^{*\gamma} + \tag{3.2}$$
$$\frac{1}{2}\rho\left(E^{*\gamma}\frac{\partial F}{\partial E^{*\gamma}} + B^{*\gamma}\frac{\partial F}{\partial B^{*\gamma}}\right)\delta_\alpha{}^\beta + \tau_\alpha^{*\beta}$$

$$R_\alpha = \rho_e E_\alpha + \frac{1}{c}\varepsilon_{\alpha\beta\gamma} j^\beta B^\gamma + \frac{\rho}{4\pi c}\frac{d}{dt}[\varepsilon^{\alpha\beta\gamma}(D_\beta B_\gamma - E_\beta H_\gamma)] +$$
$$\frac{1}{8\pi}(D_\beta\nabla_\alpha E^\beta - E^\beta\nabla_\alpha D_\beta + B_\beta\nabla_\alpha H^\beta - H_\beta\nabla_\alpha B^\beta) -$$
$$\frac{\rho}{c^2}\frac{d^*}{dt}\left[\frac{d^*\pi^{*\nu}}{dt}\frac{\partial F}{\partial(\nabla^*_\gamma\pi^{*\nu})}\right] g_{\gamma\alpha}$$

$$D^{*\alpha} = E^{*\alpha} - 4\pi\rho\frac{\partial F}{\partial E^{*\beta}} g^{\beta\alpha}, \quad H_\gamma{}^* = B_\gamma{}^* + 4\pi\rho\frac{\partial F}{\partial B^{*\gamma}}$$

$$D^\alpha = D^{*\alpha} + \varepsilon^{\alpha\beta\gamma} v_\beta H_\gamma{}^*, \quad H_\gamma = H_\gamma{}^* - \varepsilon_{\gamma\alpha\beta} v^\alpha D^{*\beta}$$

In writing the system of equations the dimensionless parameters v^2 / c^2, $p / \rho c^2$, $H^2 / \rho c^2$ and $D^2 / \rho c^2$ are, in accordance with the specified accuracy, taken as negligibly small in comparison with unity. The last two of formulas (3.2) show that the polarization and magnetization of the medium unit mass in the attached system of coordinates, whose components are, respectively, $P_\alpha{}^*$ and $M_\alpha{}^*$, are defined by formulas

$$P_\alpha{}^* = -\rho\partial F / \partial E^{*\alpha}, \quad M_\alpha{}^* = -\rho\partial F / \partial B^{*\alpha}$$

If among the arguments of the free energy F the components of the vector of magnetic induction $B^{*\alpha}$, are omitted, the magnetization in the attached coordinate system is always zero, while the independence of function F from components of the vector of the electric field intensity $E^{*\alpha}$ corresponds only to a magnetizable medium.

The functional δW in (2.1) is determined by formula

$$\delta W = \int_{\Sigma_3} [(P_\alpha{}^\beta N_\beta + P_\alpha{}^4 N_4)\delta x^\alpha - (S_\alpha{}^\beta N_\beta + S_\alpha{}^4 N_4)\delta x^\alpha - \tag{3.3}$$

$$\frac{1}{4\pi}\varepsilon^{\alpha\beta\gamma}H_\gamma N_\beta\delta_L A_\alpha + \frac{1}{4\pi c}D^\alpha N_4\delta_L A_\alpha + \frac{1}{4\pi}D^\alpha N_\alpha\delta_L\varphi + (\Psi_\sigma{}^\beta N_\beta +$$
$$\Psi_\sigma{}^4 N_4)\,\delta\pi^{*\sigma}\Big]\, d\sigma_3$$

$$P_\alpha{}^\beta = p_\alpha^\beta - \rho v_\alpha v^\beta + \rho c^{-2}\frac{\partial F}{\partial(\nabla_\nu{}^*\pi^{*\varkappa})}\frac{d^*\pi^{*\varkappa}}{dt}v^\beta g_{\nu\alpha} +$$
$$\frac{1}{4\pi c}\varepsilon_{\alpha\varkappa\gamma}v^\beta(D^\varkappa B^\gamma - E^\varkappa H^\gamma)$$

$$P_\alpha{}^4 = -\rho v_\alpha + c^{-2}v^\beta p_{\alpha\beta} + \frac{v^\beta}{2c^2}(E_\beta D_\alpha + E_\alpha D_\beta) -$$
$$\frac{\rho}{2c^2}\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\nu})}(v^\gamma g_{\alpha\beta}\nabla_\gamma{}^*\pi^{*\nu} - v_\alpha\nabla_\beta{}^*\pi^{*\nu}) +$$
$$\frac{\rho}{c^2}\frac{\partial F}{\partial(\nabla_\beta\pi^{*\nu})}\frac{d^*\pi^{*\nu}}{dt}g_{\beta\alpha}, \qquad S_\alpha{}^4 = \frac{1}{4\pi c}\varepsilon_{\alpha\beta\gamma}E^\beta H^\gamma$$

$$S_\alpha{}^\beta = -(4\pi)^{-1}(E_\alpha D^\beta + H_\alpha B^\beta) + (8\pi)^{-1}(E_\gamma D^\gamma + B_\gamma H^\gamma)\delta_\alpha{}^\beta$$

$$\Psi_\sigma{}^\beta = \rho\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\sigma})}, \qquad \Psi_\sigma{}^4 = \rho\frac{\partial F}{\partial(d^*\pi^{*\sigma}/dt)} + \frac{\rho}{c^2}v_\beta\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\sigma})}$$

where $d\sigma_3$ is an element of the three-dimensional surface Σ_3, and N_β and N_4 are the directional cosines of the four-dimensional normal to surface Σ_3, which in the space x^α, $t = x^4$ is defined by the equation $f(x^\alpha, t) = 0$. The cosines N_α and N_4 are determined by formulas

$$N_\alpha = \nabla_\alpha f \,/\, \sqrt{\nabla_\beta f\nabla^\beta f + (\partial f/\partial t)^2}$$
$$N_4 = (\partial f\,/\,\partial t)\,/\,\sqrt{\nabla_\beta f\nabla^\beta f + (\partial f\,/\,\partial t)^2}$$

Close to surface Σ_3 the inequalities $f < 0$ and $f > 0$ are valid, respectively, for internal and external points.

The requirement for the symmetry of the spatial part of the field and medium energy-momentum tensor impose additional conditions on the form of function F. In the attached system of coordinates the components of that part of the tensor are $S^{*\alpha\beta} + p^{\alpha\beta}$. This condition was taken into account in the derivation of formulas (3.1) and (3.2). If that condition is omitted, it becomes necessary to consider the equation of moments which define the variation of internal moments of momentum for the medium [4].

The equation of energy is obtained from the variational equation (2.1) after the substitution of actual increments for variations of related parameters. Transformed to a differential equation of heat influx that equation in the absence of dependence of free energy on derivatives of vector π^* is of the form

$$\rho\frac{dU}{dt} = \left(p^{\alpha\beta} - \frac{1}{2}E_\gamma{}^*P^{*\gamma}g^{\alpha\beta}\right)\nabla_\alpha{}^*v^\beta + \rho\frac{d^*}{dt}\left[\frac{\partial F}{\partial(d^*\pi^{*\alpha}/dt)}\frac{d^*\pi^{*\alpha}}{dt}\right] + \tag{3.4}$$
$$\nabla_\beta{}^*\left[\rho\frac{\partial F}{\partial(\nabla_\beta{}^*\pi^{*\alpha})}\frac{d^*\pi^{*\alpha}}{dt}\right] - P_\alpha{}^*\frac{dE^{*\alpha}}{dt} - M_\alpha{}^*\frac{dB^{*\alpha}}{dt} + E_\alpha{}^*i^{*\alpha} + \frac{dQ^e}{dt}$$

which is the same as that presented in [4]. In this equation $U = F + TS$ is the internal energy of a unit of mass.

The system is closed if, for instance, after the fixing of rheological relationships that link thermodynamic forces $\tau^{*\alpha\beta}$, $E_\alpha{}^*$, $Q_\alpha{}^*$, $-q^{*\alpha} / T^2$ with thermodynamic fluxes $\nabla_\alpha{}^* v_\beta$, $i^{*\alpha}$, $d^*\pi^{*\alpha} / dt$ and $\nabla_\alpha{}^* T$, and possibly with other parameters.

4. Model of an isotropic viscous compressible linearly-polarizable fluid. In what follows is assumed that below the components of the magnetic induction vector determined in the attached reference system do not contain among the arguments the free energy and, consequently, that the considered medium is only polarizable but not magnetizable. We denote by F_0 the free energy when all polarizations and their derivatives are zero.

It is possible to represent the free energy of a unit of mass in the form

$$F = F_0 + F_1 (\varepsilon_{\alpha\beta}, E^\alpha, T, g_{\alpha\beta}, K_\beta, \pi^\alpha, d\pi^\alpha / dt, \nabla_\beta \pi^\alpha)$$

where F_0 is independent of vectors and their derivatives which define the irreversible polarization, and also of the electric field tension, while $F_1 = 0$ when 0, $E^\alpha = 0$, $\nabla_\beta \pi^\alpha$ $\nabla_\beta \pi^\alpha = 0$, $\pi^{\cdot\alpha} = d\pi^\alpha / dt = 0$. Here and in what follows the asterisk at components determined in the attached system is everywhere omitted, since the related quantities determined in the observer's system are no longer used. We shall represent the function in the linear theory by a homogeneous quadratic form in components of vectors $\mathbf{E}$ and $\boldsymbol{\pi}$, and in temporal and spatial derivatives of the latter.

Since below we consider the case of an isotropic fluid, hence only $g_{\alpha\beta}$ and the scalar parameters ρ, T, K_B, and also E^α, π^α, $\pi^{\cdot\alpha}$ and $\nabla_\beta \pi^\alpha$ remain among the arguments of F_1 which, then, assumes the form

$$\begin{aligned} F_1 = {}& {}^1/_2 (\varkappa_1 \pi^2 - \sigma_1 \pi^{\cdot\alpha} \pi^{\cdot\beta} g_{\alpha\beta} - \varkappa_0 E^2) - \\ & \varkappa_2 E_\alpha \pi^\alpha + \sigma_2 \pi^{\cdot\alpha} \pi_\alpha + \sigma_3 E_\alpha \pi^{\cdot\alpha} + \\ & {}^1/_2 [\nu_1 \nabla_\alpha \pi^\beta \nabla^\alpha \pi_\beta + \nu_2 \nabla_\alpha \pi^\beta \nabla_\beta \pi^\alpha_\cdot + \nu_3 (\operatorname{div} \boldsymbol{\pi})^2] \end{aligned} \tag{4.1}$$

where all coefficients may be functions of density and temperature.

In the linear theory we specify a linear dependence of thermodynamic forces on thermodynamic fluxes.

The assumption of the fluid isentropicity reduces the number of phenomenological coefficients; it can be further reduced by using the Onsager symmetry relationships. From formulas which define corresponding relationships we obtain for components of the generalized force Q_α an expression of the form

$$Q_\alpha = \rho (\lambda \pi^{\cdot\beta} g_{\alpha\beta} + \lambda_1 i_\alpha + \lambda_2 \nabla_\alpha T)$$

where the coefficients λ, λ_1 and λ_2 are assumed to be known functions of ρ and T.

The equation of state for polarization, formulas that define parameter $\boldsymbol{\pi}$, the expression for stress tensor components, and the equation of heat influx are in that case of the form

$$\begin{aligned} P_\alpha &= \rho (\varkappa_0 E_\alpha + \varkappa_2 \pi_\alpha - \sigma_3 \pi^{\cdot\beta} g_{\alpha\beta}) \\ \varkappa_2 E_\alpha &= \varkappa_1 \pi_\alpha + (\sigma_2 + \lambda) \pi^{\cdot\beta} g_{\alpha\beta} - \\ & \frac{d}{dt} (\sigma_2 \pi_\alpha + \sigma_3 E_\alpha - \sigma_1 \pi^{\cdot\beta} g_{\alpha\beta}) - \rho^{-1} \nabla_\beta Y_\alpha{}^\beta \\ p_\alpha{}^\beta &= -(p_0 + \rho^2 \partial F_1 / \partial\rho) \delta_\alpha{}^\beta + {}^1/_2 E_\gamma P^\gamma \delta_\alpha{}^\beta - Y^{\beta\gamma} \nabla_\alpha \pi_\gamma \\ \rho dU_0 / dt &= -p_0 d (1 / \rho) / dt + \tau^{\alpha\beta} \nabla_\beta v_\alpha + \\ & \rho Q_\alpha \pi^{\cdot\alpha} + E_\alpha i^\alpha + \rho T d (\partial F_1 / \partial T) / dt \end{aligned} \tag{4.2}$$

$$p_0 = \rho^2 \partial F_0 / \partial \rho, \quad U_0 = F_0 - T \partial F_0 / \partial T$$
$$Y_\alpha{}^\beta = \rho\, (\nu_1 \nabla^\beta \pi_\alpha + \nu_2 \nabla_\alpha \pi^\beta + \nu_3 \operatorname{div} \pi \delta_\alpha{}^\beta)$$

where the derivatives of function F_1 with respect to ρ and T have not been expanded in order not to encumber the formula. The system of equations must be supplemented by the equation of continuity of mass, which was taken into consideration in the variation of density in Eq. (2.1). When $\sigma_3 = 0$ and $\varkappa_2 = 1$, the parameter π represents the irreversible component of the polarization vector. However, if $\varkappa_2 \neq 0$, it is always possible to substitute the new independent parameter $\pi' = \varkappa_2 \pi$ for π and reduce coefficient $\varkappa_2$ to unity.

5. Averaged formulas for linearly polarizable fluid in the case of a high-frequency nearly monochromatic field. To avoid complicating the analysis by secondary effect it is assumed below that volume charge and conduction currents are absent. We consider the case of a high-frequency field, which means that $\omega t' \gg 1$, where ω is the field frequency and t' is the characteristic time of the problem. We select a linear model and restrict the investigation to the case of fluid considered in Sect. 4. We also assume that the generalized force Q_α is independent of the temperature gradient ($\lambda_2 = 0$). In that case it is possible, as shown by formulas (3.1) and (4.2), to seek solutions for which the electric field intensity is of the form $E^\alpha = \operatorname{Re}\,[E_0{}^\alpha\,(x^\beta, t) e^{-i\omega t}]$, where $i = \sqrt{-1}$, and $E_0{}^\alpha$ is, generally speaking, a complex function which slowly varies in time. All other electromagnetic parameters B^α, π^α, D^α and P^α are similarly defined. Functions which slowly vary with time are denoted by a zero subscript.

From Eqs. (4.2) we have the following relationships:

$$\begin{gathered} P_{0\alpha} = \rho\, [\varkappa_0 E_{0\alpha} + (\varkappa_2 + \sigma_3 i\omega) \pi_{0\alpha}] \\ \varkappa_2 E_{0\alpha} = (\varkappa_1 - \lambda i\omega - \sigma_1 \omega^2) \pi_{0\alpha} + \sigma_3 i\omega E_{0\alpha} - \\ \rho^{-1} \nabla_\beta\, [\rho\, (\nu_1 \nabla^\beta \pi_{0\alpha} + \nu_2 \nabla_\alpha \pi_0{}^\beta + \nu_3 \mathrm{div} \pi_0 \delta_\alpha{}^\beta)] \end{gathered} \tag{5.1}$$

The derived formuals are based on the premise that derivatives of slowly changing quantities can be neglected because of their smallness in comparison with derivatives of quantities varying at high frequencies. They may be, however, neglected only when the first term in the right-hand side of the second of equalities (5.1) is considerably smaller than the rejected term which contains derivatives of $\pi_0{}^\alpha$ with respect to time. The constraint on the frequency necessary for eliminating frequencies close to resonance is readily formulated. When condition $(d\pi_0 / dt) / \pi_0 \ll |\varkappa_1 / \lambda|$, $\lambda^2 > |\sigma_1 \varkappa_1| \times (2^{1/2} + 1)$ is satisfied, the required inequality holds for any frequency ω. If that condition is not satisfied, terms containing derivatives of E_0 with respect to time appear in subsequent formulas.

Formulas (5.1), as well as Maxwell equations which in the considered case can be simplified by the known procedure [8], must be supplemented by thermomechanical equations in which all rapidly oscillating electromagnetic quantities varying at frequency 2ω are replaced by their time averages which will be denoted by angle brackets. It is assumed that the averaging interval is sufficiently wide to contain many oscillations of the electromagnetic quantities and, yet, reasonably small so that thermomechanical quantities and the complex amplitudes of electromagnetic quantities denoted by zero subscript remain virtually constant. Thus, for instance, for the mean square value of intensity we have the formula $\langle \mathbf{E}^2 \rangle = E_{0\alpha}\bar{E}_0{}^{\alpha} / 2$ (the stroke above a symbol denotes a complex conjugate quantity).

In the remaining part of this Section we assume the independence of free energy of spatial polarization gradients, and set coefficients σ_2 and σ_3 equal zero. In that case we have the formulas

$$P_{0\alpha} = \rho\,[\varkappa_0 + \varkappa_2{}^2 / (\varkappa_1 - \lambda i\omega - \sigma_1\omega^2)] E_{0\alpha} \tag{5.2}$$
$$D_{0\alpha} = \varepsilon E_{0\alpha}, \quad \varepsilon = 1 + 4\pi\rho\,[\varkappa_0 + \varkappa_2{}^2 / (\varkappa_1 - \lambda i\omega - \sigma_1\omega^2)]$$

where ε is the permittivity of the medium. Formulas (5.2) are known as the simplest for defining the dependence of permittivity on frequency.

The object of the described constraints is the derivation of averaged relationships which correspond to the known simplest dependence of polarization on the electric field intensity (5.2), while their omission would result in very cumbersome formulas. It should be noted that the admission of a considerable number of independent internal parameters makes it possible to obtain linearly-polarizable media with a diverse dependence of permittivity on frequency. For this it is sufficient to specify, as in [2], for the free energy a dependence of quadratic form on the field intensity, internal parameters and their derivatives with respect to time, and have the generalized forces which define the irreversibility of the polarization process represented by linear functions of derivatives of internal parameters with respect to time.

In the particular case considered here the formulas that define the averaged values of components of the total stress tensor in the medium and the dissipation due to the polarization process irreversibility are of the form

$$\langle p_\alpha{}^\beta \rangle = -\rho^2 \frac{\partial F_0}{\partial \rho} + \langle \tau_\alpha{}^\beta \rangle + \frac{\rho}{8\pi}\left[\frac{\partial \varepsilon_\infty}{\partial \rho} + \frac{|\varepsilon - \varepsilon_\infty|^2}{(\varepsilon_0 - \varepsilon_\infty)^2}\frac{\partial(\varepsilon_0 - \varepsilon_\infty)}{\partial \rho} + \right. \tag{5.3}$$
$$\left. \omega^2 |\varepsilon - \varepsilon_\infty|^2 \frac{\partial}{\partial \rho}\left(\frac{\sigma_1}{4\pi\rho\varkappa_2{}^2}\right)\right] \langle \mathbf{E}^2 \rangle\, \delta_\alpha{}^\beta$$
$$\langle \sigma \rangle = \langle Q_\alpha \pi^{\cdot\alpha} \rangle = \omega\, \mathrm{Im}\,\varepsilon \langle \mathbf{E}^2 \rangle / 4\pi$$

where ε_0 and ε_∞ are values of permittivity when ω is equal zero or infinity, respectively.

The equations of heat influx assume the form

$$\rho\frac{dU_0}{dt} = -p_0 \frac{d(1/\rho)}{dt} + \langle \tau^{\alpha\beta} \rangle \nabla_\beta v_\alpha + \frac{\omega}{4\pi}\mathrm{Im}\,\varepsilon \langle \mathbf{E}^2 \rangle - \tag{5.4}$$
$$\frac{\rho T}{8\pi}\frac{d}{dt}\left\{\frac{1}{\rho}\left[\frac{\partial \varepsilon_\infty}{\partial T} + \omega^2 |\varepsilon - \varepsilon_\infty|^2 \frac{\partial}{\partial T}\left(\frac{\sigma_1}{4\pi\rho\varkappa_2{}^2}\right) + \right.\right.$$
$$\left.\left. \frac{|\varepsilon - \varepsilon_\infty|^2}{(\varepsilon_0 - \varepsilon_\infty)^2}\frac{\partial(\varepsilon_0 - \varepsilon_\infty)}{\partial T}\right] \langle \mathbf{E}^2 \rangle\right\}$$

The second of formulas (5.3) is valid in the linear theory for very general assumptions and is widely known (see, e.g., [8]). The first of formulas (5.3) and formula (5.4) are of a less general character, since they are related to specific assumptions on the form of free energy and on closing rheological relationships. In the absence of irreversibility which is associated with the polarization process, the expression for stress tensor components can be reduced to the formula obtained in [9] with (5.2) taken into account.

The formula for components of the time averaged volume force $\langle R_\alpha \rangle$ is of the form

$$\langle R_\alpha\rangle = (4c)^{-1}\frac{d}{dt}[\varepsilon_{\alpha\beta\gamma}(\overline{B}_0{}^\beta P_0{}^\gamma + B_0{}^\beta\overline{P}_0{}^\gamma)] + \tag{5.5}$$
$$({}^1/_8)(P_0{}^\gamma\nabla_\alpha\overline{E}_{0\gamma} + \overline{P}_0{}^\gamma\nabla_\alpha E_{0\gamma} - \overline{E}_{0\gamma}\nabla_\alpha P_0{}^\gamma - E_{0\gamma}\nabla_\alpha\overline{P}_0{}^\gamma)$$

6. Averaged relationships that define the interaction between the field and medium in the geometrical optics approximation. Since we have in mind the case of propagation in the medium of a nearly monochromatic light wave, we shall assume that the inequality $\omega L / c \gg 1$, where L is a characteristic linear dimension of the problem, is satisfied besides the inequality $\omega t' \gg 1$. As previously, we assume that the coefficient λ_2 is zero, and seek the electric field intensity in the form

$$E_0{}^\beta(x^\alpha, t) = E_{00}{}^\beta(x^\alpha, t)\exp[i(\omega / c)\zeta(x^\alpha, t)] \tag{6.1}$$

where, generally speaking, the complex functions $E_{00}{}^\beta$ and ζ slowly vary in time and with respect to coordinates. All remaining electromagnetic quantities are represented by similar formulas, with the related complex amplitudes denoted by double-zero subscripts.

The substitution of formula (6.1) and of similar formulas for other electromagnetic quantities into (5.1) yields the following equations:

$$E_{00\alpha} = A_{\alpha\beta}\pi_{00}{}^\beta, \quad D_{00}{}^\alpha = \sigma^{\alpha\beta}E_{00\beta} \tag{6.2}$$
$$A_{\alpha\beta} = \varkappa_1{}^{-1}\{[\varkappa_2 - \lambda i\omega - \sigma_1\omega^2 + \nu_1(\omega / c)^2\nabla_\gamma\zeta\nabla^\gamma\zeta]g_{\alpha\beta} +$$
$$(\nu_2 + \nu_3)(\omega / c)^2\nabla_\alpha\zeta\nabla_\beta\zeta\}$$

(for simplicity it is assumed here, as in Sect. 5, that $\sigma_2 = \sigma_3 = 0$)

Although the meaning of $\sigma^{\alpha\beta} = \rho[(1 + 4\pi\varkappa_0)g^{\alpha\beta} + 4\pi\varkappa_2 A^{-1\alpha\beta}]$, where $A^{-1\alpha\beta}A_{\beta\gamma} = \delta^\alpha_\gamma$, is that of the medium tensor permittivity, it also links the complex amplitudes denoted in (6.2) by double-zero subscripts. Note that optical anisotropy may appear owing to the nonzero coefficient $\nu_2 + \nu_3$ also in the case of the model of fluid which is isotropic and linear with respect to polarization, when the dependence on the polarization gradient is taken into account. In the derivation of formulas (6.2) the derivatives of slow varying quantities taken with respect to coordinates were neglected because of their smallness as compared with those of quantities that vary rapidly with respect to coordinates.

The substitution of formula (6.1) and of the similar formula for the magnetic field intensity into Maxwell equations yield in accordance with the method of geometrical optics [10] the following asymptotic formulas in terms of the large parameter $\omega L / c$:

$$|\chi^{\alpha\beta\gamma\varkappa}\nabla_\beta\zeta\nabla_\gamma\zeta + \sigma^{\alpha\varkappa}| = 0 \tag{6.3}$$
$$\chi^{\alpha\beta\gamma\varkappa}[\nabla_\beta\zeta\nabla_\gamma E_{00\varkappa} + \nabla_\beta(\nabla_\gamma\zeta E_{00\varkappa})] - 2i\omega c^{-2}\sigma^{\alpha\gamma}E_{00\gamma}\,\partial\zeta / \partial t = 0$$
$$\chi^{\alpha\beta\gamma\lambda\mu\nu}[\nabla_\beta\zeta\nabla_\mu H_{00\nu}\sigma^{-1}_{\gamma\lambda} + \nabla_\beta(\nabla_\mu\zeta H_{00\nu}\sigma^{-1}_{\gamma\lambda})] - 2i\omega c^{-2}H^\alpha_{00}\,\partial\zeta / \partial t = 0$$
$$\chi^{\alpha\beta\gamma\lambda\mu\nu} = \begin{vmatrix} g^{\alpha\lambda} & g^{\alpha\mu} & g^{\alpha\nu} \\ g^{\beta\lambda} & g^{\beta\mu} & g^{\beta\nu} \\ g^{\gamma\lambda} & g^{\gamma\mu} & g^{\gamma\nu} \end{vmatrix}$$
$$\chi^{\alpha\beta\gamma\varkappa} = g^{\alpha\beta}g^{\gamma\varkappa} - g^{\alpha\varkappa}g^{\beta\gamma}, \qquad \sigma^{-1}_{\alpha\beta}\sigma^{\beta\gamma} = \delta_\alpha{}^\gamma$$

The first of these is the equation for determining function $\zeta(x^\alpha, t)$, and the remaining make possible to determine the quantities $E_{00\varkappa}$ and $H_{00\nu}$. In the general case, when the dependence on coordinates and time is expressed in terms of known thermomechanical parameters, Eqs. (6.3) link with other equations and replace Maxwell equations. Further simplification is achieved by averaging thermomechanical variables over three-dimensional volumes that are small in comparison with L but contain many wavelengths (the wavelength order of magnitude is $c / \omega \ll L$).

Owing to the cumbersomeness of the complete system of equations it is not presented here, and only the case in which the free energy is independent of gradients of vector π is considered. Formulas (5.2) — (5.4) remain valid, if the subscript 00 is substituted for 0 and the mean square of intensity is taken as its mean with respect to time and volume, which is calculated by formula

$$\langle \mathbf{E}^2 \rangle = {}^1/_2 E_{00\alpha} \bar{E}_{00}{}^\alpha \exp(-2 \operatorname{Im} \zeta \omega / c)$$

After averaging over the volume and estimating the order of its terms, formula (5.5) is further simplified and reduced to the form

$$\langle R_\alpha \rangle = (8\pi)^{-1} [-\langle E^2 \rangle \nabla_\alpha (\operatorname{Re} \varepsilon) + 2(\omega / c) \operatorname{Im} \varepsilon \langle E^2 \rangle \nabla_\alpha (\operatorname{Re} \zeta)]$$

7. On the description of the Kerr effect. The model considered here also admits a nonquadratic dependence of free energy on the field intensity and other electrodynamic parameters. In particular, it is possible to describe the electro-optical Kerr effect with the use of function F that is nonquadratic with respect to E^α. For this it is necessary to consider a solution of the form $E_\alpha = E_{1\alpha} + E_{0\alpha} e^{-i\omega t}$, where $E_{1\alpha}$ and $E_{0\alpha}$ are slowly varying functions of time. Let the free energy be defined by $F = F_0 + F_1 (E^\alpha, \rho, T, g_{\alpha\beta})$, where F_0 is independent of variables of the electromagnetic character, $Q_\alpha = 0$, and F_1 is defined by the expression $F_1 = -(\alpha_0 \mathbf{E}^2 / 2 + \alpha_1 \mathbf{E}^4 / 4)$ on the assumption of fluid isotropicity. Then, if the high-frequency field intensity component E_0 is considerably lower than the slow varying component E_1, it is possible to restrict the expansion of vector $\mathbf{D}$ in components of vector $\mathbf{E}_0$ to its zero and first terms. We have

$$D_\alpha = D_{1\alpha} + D_{0\alpha} e^{-i\omega t}$$

$$D_{1\alpha} = (\varepsilon_0 + 4\pi\alpha_1 \rho E_1{}^2) E_{1\alpha}, \quad D_{0\alpha} = \sigma_{\alpha\beta} E_0{}^\beta$$

$$\varepsilon_0 = 1 + 4\pi\alpha_0 \rho, \quad \sigma_{\alpha\beta} = (\varepsilon_0 + 4\pi\alpha_1 \rho \mathbf{E}_1{}^2) g_{\alpha\beta} + 8\pi\alpha_1 \rho E_{1\alpha} E_{1\beta}$$

The last term of the formula for permittivity defines the Kerr effect [8].

The author thanks L. I. Sedov for guidance and A. G. Tsypkin for valuable remarks.

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Translated by J. J. D.